

# DOMAINS WITH INVERTIBLE-RADICAL FACTORIZATION

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**ABSTRACT.** We study those integral domains in which every proper ideal can be written as an invertible ideal multiplied by a nonempty product of proper radical ideals.

In [15] Vaughan and Yeagy introduced and studied the notion of *SP-domain*, i.e. an integral domain whose ideals are products of radical (also called semiprime) ideals. They proved that an SP-domain is always almost Dedekind (i.e. every localization at a maximal ideal is a rank one discrete valuation domain (DVR)). They also gave an example of an SP-domain which is not Dedekind. For examples of almost Dedekind domains which are not SP, see [16] and [6, Example 3.4.1]. The study of SP-domains was continued by Olberding (in [10]) who gave several characterizations for SP-domains inside the class of almost Dedekind domains and also gave a method to construct SP-domains starting from Boolean topological spaces.

In a sequence of papers ([11], [12], [13]) Olberding introduced and studied the concept of *ZPUI* (*Zerlegung Prim und Umkehrbaridealen*) domain, i.e. a domain for which every proper nonzero ideal can be factored as a product of an invertible ideal times a nonempty product of (pairwise comaximal) prime ideals (Olberding did his study for commutative rings, but we are interested here only in domain case). He showed that a domain  $A$  is ZPUI if and only if every proper nonzero ideal can be factored as a product of a finitely generated ideal times a nonempty finite product of prime ideals if and only if  $A$  is a strongly discrete h-local Prüfer domain [13, Theorem 1.1]. Let  $A$  be a domain. We recall that  $A$  is *h-local* if the factor ring  $A/I$  is local (resp. semilocal) for each nonzero prime ideal (resp. nonzero ideal)  $I$  of  $A$ . Also  $A$  is a *Prüfer domain* if its non-zero finitely generated ideals are invertible. A Prüfer domain is *strongly discrete* if it has no idempotent prime ideal except zero.

In this paper we introduce a class of domains having the classes of SP-domains (resp. ZPUI domains) as subclasses. Call a domain  $A$  an *ISP-domain* (*invertible semiprime domain*) if each proper ideal of  $A$  can be written as an invertible ideal multiplied by a nonempty product of proper radical ideals. So an ISP-domain need not be one-dimensional or h-local.

In Section 1 we prove the following results. A valuation domain is an ISP-domain if and only if it is strongly discrete (Proposition 2). If  $A$  is an ISP-domain, then any factor domain of  $A$  and any (flat) overring of  $A$  are also ISP-domains (Propositions 4 and 5, see also Proposition 15). Any one-dimensional ISP-domain is almost Dedekind and, consequently, any Noetherian ISP-domain is a Dedekind domain (Proposition 6 and Corollary 7). In Section 2 we prove that if  $A$  is an ISP-domain, then  $A$  is a strongly discrete Prüfer domain and every nonzero prime

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ideal of  $A$  is contained in a unique maximal ideal (Theorem 18). Consequently, an ISP-domain such that every ideal has finitely many minimal prime ideals is a ZPUI-domain (Proposition 16). In Section 3 we consider the question whether every one-dimensional ISP-domain is an SP-domain. We provide a positive answer for domains in which every nonzero element is contained in at most finitely many noninvertible maximal ideals (Theorem 21). In particular, a one-dimensional ISP-domain having only finitely many noninvertible maximal ideals is an SP-domain (Corollary 22). In Section 4 we give an example of a two-dimensional ISP-domain  $A$  which is not h-local. Hence  $A$  is neither an SP-domain nor a ZPUI-domain.

Throughout this paper, our rings are commutative and unitary. For any undefined terminology, we refer the reader to [8] or [9].

## 1. BASIC RESULTS

We recall the key definition of our paper.

**Definition 1.** We say that a domain  $A$  is an *ISP-domain* (*invertible semiprime domain*) if every proper nonzero ideal  $I$  of  $A$  can be written as  $JQ_1 \cdots Q_n$  where  $n \geq 1$ ,  $J$  is an invertible ideal and each  $Q_i$  is a proper radical ideal.

Clearly a ZPUI-domain or an SP-domain is an ISP-domain. So an ISP-domain need not be one-dimension or h-local. In this section we derive some basic properties of SP-domains.

**Proposition 2.** *For a valuation domain  $A$ , the following are equivalent:*

- (a)  $A$  is an ISP-domain.
- (b) Every nonzero proper ideal  $I$  can be written as  $xP^n$  for some  $x \in A$ ,  $n \geq 1$  and some prime ideal  $P$ .
- (c)  $A$  is strongly discrete.

*Proof.* (a)  $\Leftrightarrow$  (b) As  $A$  is a valuation domain, all radical ideals of  $A$  are prime and, for  $P \subseteq Q$  prime ideals, we have  $PQ = P$ . (b)  $\Leftrightarrow$  (c) is [3, Theorem 2].  $\square$

**Example 3.** The Bezout domain  $A = \mathbb{Z} + X\mathbb{Q}[X]$  is not ISP. Indeed, consider the ideal  $I = X\mathbb{Z}[1/2] + X^2\mathbb{Q}[X]$ . It is easy to see that the radical ideals containing  $I$  are  $X\mathbb{Q}[X]$  and  $nA = n\mathbb{Z} + X\mathbb{Q}[X]$  with  $n$  a positive square-free integer. So there is no element  $f \in A$  such that  $I \subseteq fA$  and  $If^{-1}$  is a product of radical ideals. Note that every proper nonzero principal ideal  $gA$  can be written in the form required by Definition 1. Indeed, if  $g \notin X\mathbb{Q}[X]$ , then  $g$  is a product of principal primes and if  $g \in X\mathbb{Q}[X]$ , then  $g = 2(g/2)A$ . Note also that  $A$  is strongly discrete. Some basic properties of  $A$  can be found in [4].

**Proposition 4.** *If  $A$  is an ISP-domain and  $P$  a prime ideal of  $A$ , then  $A/P$  is an ISP-domain.*

*Proof.* Let  $I \supset P$  be a proper ideal of  $A$ . As  $A$  is an ISP-domain, we can write  $I = JH_1 \cdots H_n$  with  $J$  an invertible ideal,  $n \geq 1$  and each  $H_i$  a proper radical ideal. Since all ideals  $I, H_1, \dots, H_n$  contain  $P$ , we get  $I/P = (J/P)(H_1/P) \cdots (H_n/P)$  with  $J/P$  invertible and each  $H_i/P$  a proper radical ideal.  $\square$

**Proposition 5.** *Let  $A$  be an ISP-domain and  $B$  a flat overring of  $A$ . Then  $B$  is an ISP-domain.*

*Proof.* Let  $H$  a proper nonzero ideal of  $B$  and  $I = H \cap A$ . By [2, Theorem 2],  $IB = H$ . As  $A$  is an ISP-domain, we can write  $I = JQ_1 \cdots Q_n$  with  $J$  an invertible ideal,  $n \geq 1$  and all  $Q_i$ 's proper radical ideals. Then  $H = IB = (JB)(Q_1B) \cdots (Q_nB)$ , where  $JB$  is invertible and each  $Q_iB$  is a radical ideal. Indeed, since  $A_{M \cap A} = B_M$  for every  $M \in \text{Max}(B)$  (cf. [2, Theorem 2]), it is easy to check locally that a radical ideal of  $A$  extends to a radical ideal of  $B$ . If every  $Q_iB$  is equal to  $B$ , then  $H = JB$  and  $WB = B$  where  $W = Q_1 \cdots Q_n$ . Hence  $J \subseteq JB \cap A = H \cap A = I = JW \subseteq J$ , so  $J = JW$ , thus  $W = A$  (because  $J$  is invertible), a contradiction.  $\square$

**Proposition 6.** *Any one-dimensional ISP-domain is almost Dedekind.*

*Proof.* Let  $A$  be a one-dimensional ISP-domain. By Proposition 5, we may assume that  $A$  is local with maximal ideal  $M$ . Let  $x \in M - \{0\}$ . Since the radical ideals of  $A$  are 0 and  $M$ , we get  $xA = yM^k$  for some  $y \in A$  and  $k \geq 1$ , so  $M$  is invertible, hence  $A$  is a DVR.  $\square$

Clearly, a Dedekind domain is a Noetherian ISP-domain. The converse is also true:

**Corollary 7.** *A Noetherian ISP-domain is a Dedekind domain.*

*Proof.* Assume, by the contrary, that  $A$  is a Noetherian ISP-domain which is not Dedekind. By Proposition 6,  $\dim(A) \geq 2$ , so, using Proposition 5, we may assume that  $A$  is a two-dimensional local domain (with maximal ideal  $M$ ). Let  $x \in M - M^2$ ,  $P$  a height one prime ideal containing  $x$  and let  $y \in M - P$ . Since  $P \not\subseteq M^2$ ,  $M$  is minimal over  $(P, y^2)$  and  $A$  is an ISP-domain, we get  $(P, y^2) = M$ . Modding out by  $P$ , we get a contradiction.  $\square$

## 2. ISP DOMAINS ARE PRÜFER STRONGLY DISCRETE

**Lemma 8.** *If  $A$  is an ISP-domain and  $P \subset M$  are nonzero prime ideals of  $A$ , then  $P \subseteq M^2A_M$ .*

*Proof.* By Proposition 5, we may assume that  $A$  is local with maximal ideal  $M$ . Assume that  $P \not\subseteq M^2$  and take  $x \in M - P$ . Since  $A$  is an ISP-domain and  $P \not\subseteq M^2$ , we get that  $(P, x^2)$  is a radical ideal, so  $(P, x^2) = (P, x)$  which gives a contradiction after modding out by  $P$ .  $\square$

**Lemma 9.** *Let  $A$  be an ISP-domain,  $P \subset M$  prime ideals and  $x \in M - P$  such that  $M$  is minimal over  $(P, x)$ . Then  $MA_M$  is a principal ideal.*

*Proof.* By Proposition 5, we may assume that  $A$  is local with maximal ideal  $M$ . We show first that  $M$  is not idempotent. Deny. Note that  $\sqrt{(P, x)} = M$  is the only radical ideal containing  $(P, x)$ . As  $A$  is an ISP-domain and  $M = M^2$ , we get  $(P, x) = yM$  for some  $y \in A$ . As  $P \subseteq yM$ , we get  $y \notin P$  (otherwise  $P = yA \subseteq yM$ ), hence  $P = Py$ . From  $x \in yM$ , we get  $x = yz$  for some  $z \in M$ . Now from  $(Py, yz) = yM$ , we get  $(P, z) = M$ , so  $M/P$  is a principal idempotent nonzero maximal ideal of  $A/P$ , a contradiction. Thus  $M$  is not idempotent and let us pick  $w \in M - M^2$ . By Lemma 8,  $M$  is the only prime ideal containing  $w$ , so  $wA = M$  because  $A$  is an ISP-domain.  $\square$

**Lemma 10.** *If  $A$  is an ISP-domain and  $I$  an invertible radical proper ideal of  $A$ , then  $A/I$  is zero-dimensional.*

*Proof.* Deny. Then there exist two prime ideals  $P \subset M$  and  $x \in M - P$  such that  $I \subseteq P$  and  $M$  is minimal over  $(P, x)$ . By Lemma 9,  $MA_M$  is principal. Localizing at  $M$ , we may assume that  $A$  is local with maximal ideal  $M$ . Then  $I = yA$  and  $M = zA$  for some  $y, z \in A$ . As  $I \subset M$ , we get  $y = az^2$  for some  $a \in A$ , so  $az \in \sqrt{yA} = yA$ , hence  $y = az^2 \in yzA$ , thus  $1 \in zA = M$ , a contradiction.  $\square$

We can retrieve [15, Theorem 2.4].

**Corollary 11.** *Any SP-domain  $A$  is almost Dedekind.*

*Proof.* Let  $P$  be a nonzero prime and  $x \in P - \{0\}$ . By hypothesis,  $xA = J_1 \cdots J_n$  with each  $J_i$  an invertible radical ideal. As  $P$  is prime, it contains some  $H_i$ , so  $P$  is maximal (cf. Lemma 10). Thus  $A$  is one-dimensional and Proposition 6 applies.  $\square$

**Lemma 12.** ([13, Lemma 3.2]) *A domain  $A$  is strongly discrete Prüfer if and only if  $PA_P$  is a principal ideal for every nonzero prime ideal  $P$  of  $A$ .*

**Theorem 13.** *Any ISP-domain is a strongly discrete Prüfer domain.*

*Proof.* By Proposition 5 and Lemma 12, it suffices to show that if  $A$  is a local ISP-domain, then its maximal ideal  $M$  is principal. Given  $x \in M - \{0\}$ , we write  $xA = yH_1 \cdots H_n$  with  $y \in A$ ,  $n \geq 1$  and  $H_1, \dots, H_n$  proper (principal) radical ideals. By Lemma 10, we have  $\text{Spec}(A/H_1) = \{M/H_1\}$ , hence  $H_1 = \sqrt{H_1} = M$ .  $\square$

**Corollary 14.** *A local domain is an ISP-domain if and only if it is a strongly discrete valuation domain.*

**Corollary 15.** *Any overring of an ISP-domain is also an ISP-domain.*

*Proof.* Let  $A$  be an ISP-domain and  $B$  an overring of  $A$ . By Theorem 13,  $A$  is a Prüfer domain, so  $B$  is  $A$ -flat, cf. [14, page 798]. Apply Proposition 5.  $\square$

**Proposition 16.** *For a domain  $A$ , the following assertions are equivalent.*

- (a)  *$A$  is a ZPUI-domain.*
- (b)  *$A$  is an h-local strongly discrete Prüfer domain.*
- (c)  *$A$  is an h-local ISP-domain.*
- (d)  *$A$  is a generalized Dedekind ISP-domain.*
- (e)  *$A$  is an ISP-domain such that  $\text{Min}(I)$  is finite for each ideal  $I$ .*

*Proof.* (a)  $\Leftrightarrow$  (b) is the main theorem of [13]. Implications ((a) and (b))  $\Rightarrow$  (c)  $\Rightarrow$  (d)  $\Rightarrow$  (e) are well-known. For (e)  $\Rightarrow$  (a), it suffices to show that every nonzero radical ideal  $I$  is a product of primes. Let  $\text{Min}(I) = \{P_1, \dots, P_n\}$ . Then  $I = P_1 \cap \cdots \cap P_n = P_1 \cdots P_n$  because  $P_1, \dots, P_n$  are incomparable ideals in a Prüfer domain, hence comaximal.  $\square$

By implication (e)  $\Rightarrow$  (a) above, a semilocal ISP-domain is ZPUI, hence h-local.

**Proposition 17.** *If  $A$  is an ISP-domain, then every nonzero prime ideal of  $A$  is contained in a unique maximal ideal.*

*Proof.* Deny. Let  $P$  a nonzero prime ideal and  $M_1$  and  $M_2$  two distinct maximal ideals containing  $P$ . Set  $S = A - (M_1 \cup M_2)$ . Changing  $A$  by  $A_S$  (cf. Proposition 5), we may assume that  $\text{Max}(A) = \{M_1, M_2\}$ . By remark made before this proposition, we have a contradiction.  $\square$

Consequently, we get an improved form of Theorem 13.

**Theorem 18.** *If  $A$  is an ISP-domain, then  $A$  is a strongly discrete Prüfer domain and every nonzero prime ideal of  $A$  is contained in a unique maximal ideal.*

### 3. ALMOST DEDEKIND ISP-DOMAINS

In this section, we consider the question whether any one-dimensional ISP-domain is an SP-domain. First, we recall some definitions. Let  $A$  be an almost Dedekind domain. The maximal ideals of  $A$  containing a radical invertible ideal are called *non-critical*, while the others are called *critical*. Given  $I$  an ideal of  $A$  and  $n \geq 1$ , we set  $V_n(I) = \{M \in \text{Max}(A) \mid I \subseteq M^n\}$ . Note that  $V_{n+1}(I) \subseteq V_n(I)$  and  $V_1(I)$  is the usual Zariski closed set  $V(I)$ . Next, we recall [10, Theorem 2.1] and add a new assertion (g).

**Theorem 19.** ([10, Theorem 2.1]) *For an almost Dedekind domain  $A$ , the following assertions are equivalent.*

- (a)  $A$  is an SP-domain.
- (b)  $A$  has no critical maximal ideals.
- (c) The radical of an invertible ideal is invertible.
- (d) Every principal ideal is a product of radical ideals.
- (e) For every nonzero proper (principal) ideal  $I$  and  $n \geq 1$ , the set  $V_n(I)$  is (Zariski) closed in  $\text{Spec}(A)$  and  $V_m(I)$  is empty for some large  $m$ .
- (f) Every nonzero proper ideal  $I$  can be factorized (uniquely) as  $I = J_1 J_2 \cdots J_n$  with radical ideals  $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n$ .
- (g) For every nonzero proper ideal  $I$ , we have  $I = \sqrt{I}H$  for some ideal  $H$ .

*Proof.* Since only (g) is new, it suffices to prove the equivalence of (f) and (g). (g)  $\Rightarrow$  (f) We have  $I = \sqrt{I}H_1$  and  $H_1 = \sqrt{H_1}H_2$  for some ideals  $H_1$  and  $H_2$ . Set  $J_1 = \sqrt{I}$  and  $J_2 = \sqrt{H_1}$ , so  $I = J_1 J_2 H_2$ . From  $I \subseteq H_1$ , we get  $J_1 \subseteq J_2$ . Repeating, we get  $I = J_1 J_2 \cdots J_n H_n$  with radical ideals  $J_1 \subseteq \cdots \subseteq J_n$ . If some  $H_n$  is  $A$ , we are done. If not, let  $M$  be a maximal ideal containing all  $J_i$ 's. Then  $I = J_1 J_2 \cdots J_n H_n \subseteq M^n$  for each  $n \geq 1$ , which is a contradiction because  $A_M$  is a DVR. Conversely, from  $I = J_1 \cdots J_n$  with  $J_1 \subseteq \cdots \subseteq J_n$  radical ideals, we get  $\sqrt{I} = J_1$ , so we are done.  $\square$

We recall two known facts.

**Lemma 20.** *Let  $A$  be an almost Dedekind domain which is not Dedekind. The following assertions hold.*

- (a) Every noninvertible nonzero ideal of  $A$  is contained in some noninvertible maximal ideal.
- (b) Every infinite closed subset of  $\text{Max}(A)$  contains some noninvertible maximal ideal.

*Proof.* (a) is a well-known application of Zorn's Lemma (every non finitely generated ideal can be embedded in a non finitely generated prime ideal). (b) Let  $I$  be a nonzero ideal such that  $V(I)$  is infinite. By (a), we may assume that  $I$  is invertible, so the assertion follows from [6, Proposition 3.2.2]. We give an alternative proof. For each  $P \in V(I)$ , we have  $IA_P = (PA_P)^{n_P}$  for some (unique) positive integer  $n_P$ . Consider the ideal  $H = \sum_{P \in V(I)} IP^{-n_P}$ . It suffices to show that  $H$  is not finitely generated, because  $I \subseteq H$  implies  $V(H) \subseteq V(I)$ , so part (a) applies. Suppose that  $H$  is finitely generated. Then there exist distinct ideals  $P_1, \dots, P_{k+1} \in V(I)$  such that  $IP_{k+1}^{-n_{k+1}} \subseteq \sum_{i=1}^k IP_i^{-n_i}$  where  $n_j = n_{P_j}$ . Since the ideals  $P_j$  are mutually

comaximal, we have  $IP_{k+1}^{-n_{k+1}} \subseteq I(\cap_{i=1}^k P_i^{n_i})^{-1}$ , cf. [11, Lemma 5.1]. We cancel  $I$  and get  $\cap_{i=1}^k P_i^{n_i} \subseteq P_{k+1}$ , which is a contradiction.  $\square$

Let  $A$  be an almost Dedekind domain in which every nonzero element is contained in at most finitely many noninvertible maximal ideals. Equivalently,  $A$  is an almost Dedekind domain which has weak factorization, cf. [6, Proposition 4.2.14] (recall that *having weak factorization* means that every nonzero nondivisorial ideal  $I$  factors as  $I = I_v M_1 M_2 \cdots M_n$  for some maximal ideals  $M_1, M_2, \dots, M_n$ , cf. [5]). Denote by  $Z$  the set of noninvertible maximal ideals of  $A$ . We introduce an ad-hoc concept: call an ideal  $H$  of  $A$  a *clean ideal*, if  $H$  is invertible,  $V(H) \cap Z = \{M\}$  and  $H \not\subseteq M^2$ . Let  $M \in Z$  and  $f \in M - \{0\}$ . By our hypothesis  $V(f) \cap Z$  is finite, say equal to  $\{M, M_1, \dots, M_n\}$ . By Prime Avoidance Lemma (e.g. [8, Proposition 4.9]), we can pick an element  $g \in M - (M^2 \cup M_1 \cup \cdots \cup M_n)$ , so  $(f, g)$  is clean. Hence every  $M \in Z$  contains a clean ideal. With notation above, we have:

**Theorem 21.** *Let  $A$  be an almost Dedekind domain in which every nonzero element is contained in at most finitely many noninvertible maximal ideals. The following assertions are equivalent.*

- (a)  *$A$  is an SP-domain.*
- (b)  *$A$  is an ISP-domain.*
- (c) *For every clean ideal  $H$ , the set  $V_2(H)$  is finite.*
- (d) *Every  $M \in Z$  contains a clean ideal  $H$  such that  $V_2(H)$  is finite.*

*Proof.* We may assume that  $A$  is not a Dedekind domain. Set  $F = \text{Max}(A) - Z$ . (a)  $\Rightarrow$  (b) is obvious. (b)  $\Rightarrow$  (c) Assume, to the contrary, that  $H$  is a clean ideal and  $V_2(H)$  contains an infinite set  $\{P_n \mid n \geq 1\} \subseteq F$ . Set  $V(H) \cap Z = \{M\}$ . Let  $I$  be the (integral) ideal  $\sum_{n \geq 0} HP_{2n+1}^{-1}$ . Since  $H \subseteq I$  and  $V(H) \cap Z = \{M\}$ , we get  $V(I) \cap Z = \{M\}$ , because  $M \supseteq H = P_{2n+1} HP_{2n+1}^{-1}$  implies  $M \supseteq HP_{2n+1}^{-1}$ . As  $A$  is an ISP-domain, we can write  $I = JQ$  with  $J$  an invertible ideal and  $Q \neq A$  a product of radical ideals. Since  $M \in V(I) - V_2(I)$ , we have one of the two cases below.

*Case 1:  $M \supseteq J$  and  $M \not\supseteq Q$ .* Then  $V(Q) \cap Z$  is empty, so  $Q$  is invertible, cf. Lemma 20. So  $I = JQ$  is invertible, hence finitely generated. Then  $HP_{2n+1}^{-1} \subseteq HP_1^{-1} + \cdots + HP_{2n-1}^{-1}$  for some  $n \geq 1$ . Since  $H$  can be cancelled and the other ideals involved are invertible and comaximal, we get  $P_{2n+1}^{-1} \subseteq (P_1 \cap \cdots \cap P_{2n-1})^{-1}$  (cf. [11, Lemma 5.1]), hence  $P_{2n+1} \supseteq P_1 \cap \cdots \cap P_{2n-1}$ , which is a contradiction.

*Case 2:  $M \not\supseteq J$  and  $M \supseteq Q$ .* Since  $H \subseteq Q$  and  $H \not\subseteq M^2$ , we have that  $V_2(Q) \cap Z = \emptyset$ . As  $Q$  is a product of radical ideals, [1, Lemma 1.10] shows that  $V_2(Q)$  is closed, so  $V_2(Q)$  is finite, cf. Lemma 20. Note that  $P_{2n} \in V_2(I)$  for every  $n \geq 1$ . Consequently, there exists some  $m \geq 1$  such that  $P_{2n} \in V(J)$  for each  $n \geq m$ . By Lemma 20 and the fact that  $H \subseteq J$ , we get  $V(J) \cap Z = \{M\}$ , which is a contradiction.

(c)  $\Rightarrow$  (d) is clear. (d)  $\Rightarrow$  (a) By [10, Theorem 2.1], it suffices to show that each  $M \in Z$  contains an invertible radical ideal. By (d),  $M$  contains a clean ideal  $H$  such that  $V_2(H)$  is finite, say equal to  $\{P_1, \dots, P_n\}$ . For each  $i$  between 1 and  $n$ , we have  $HA_{P_i} = P_i^{k_i} A_{P_i}$  for some  $k_i \geq 2$ . Then  $HP_1^{-k_1} \cdots P_n^{-k_n}$  is an invertible radical ideal contained in  $M$ .  $\square$

Note that [10, Example 4.3] is a SP-domain with nonzero Jacobson radical and having no finitely generated maximal ideals (hence not having weak factorization).

**Corollary 22.** *Let  $A$  be almost Dedekind domain having only finitely many noninvertible maximal ideals. Then  $A$  is an ISP-domain if and only if  $A$  is an SP-domain.*

**Corollary 23.** *Let  $A$  be an ISP-domain which has weak factorization and  $B$  a one-dimensional overring of  $A$ . Then  $B$  is an SP-domain.*

*Proof.* By Theorem 13,  $A$  is a strongly discrete Prüfer domain, so  $B$  has weak factorization, cf. [6, Corollary 4.3.3]. Now apply Corollary 15 and Theorem 21.  $\square$

The following question remains.

**Question 24.** Is every one-dimensional ISP-domain an SP-domain ?

#### 4. AN EXAMPLE

In this final section we give an example of a two-dimensional ISP-domain  $A$  which is not h-local. Hence  $A$  is neither an SP-domain nor a ZPUI-domain.

**Proposition 25.** *Let  $C$  be an SP-domain but not Dedekind,  $M = qC$  a maximal principal ideal of  $C$  and  $D$  a DVR with quotient field  $C/M$ . Assume there exists a unit  $p$  of  $C$  such that  $\pi(p)$  generates the maximal ideal of  $D$ , where  $\pi : C \rightarrow C/M$  is the canonical map. Then the pull-back domain  $A = \pi^{-1}(D)$  is a two-dimensional ISP-domain which is not h-local.*

*Proof.* As  $\pi(Mp^{-1}) = 0$ , it follows that  $M \subseteq pA$ , so  $A/pA$  is the residue field of  $D$ , because  $A/M = D$  and  $\pi(p)$  generates the maximal ideal of  $D$ . Also, the only prime ideal of  $A$  strictly containing  $M$  is the maximal ideal  $pA$ . By standard pull-back arguments (see for instance [7, Lemma 1.1.4]), the map  $P \mapsto P \cap A$  is a bijection from  $\text{Spec}(C) - V(M)$  to  $\text{Spec}(A) - V(M)$  and  $A_{P \cap A} = C_P$ . By [7, Corollary 1.1.9],  $A$  is a two-dimensional Prüfer domain. Also, by [7, Lemma 1.1.6], we have  $A[p^{-1}] = C[p^{-1}] = C$ . Roughly speaking,  $\text{Spec}(A)$  is obtained from  $\text{Spec}(C)$  by adding the maximal ideal  $pA \supseteq M$ . Since  $C$  is an almost Dedekind domain which is not Dedekind, there exists a nonzero element  $z \in A$  belonging to infinitely many maximal ideals of  $A$ , so  $A$  is not h-local. By [7, Proposition 5.3.3],  $B = A_{pA}$  is a two-dimensional strongly discrete valuation domain. It follows that  $\cap_{t \geq 1} p^t A = M$ .

Let  $I$  be an ideal of  $A$ . We observe that  $I = IB \cap IC$ . Indeed, if  $N \in \text{Max}(A) - \{pA\}$ , then  $I \subseteq IC_{A-N} = IA_N$ , so  $IB \cap IC \subseteq \cap_{Q \in \text{Max}(A)} IA_Q = I$ . In particular, we have  $A = B \cap C$ . Since  $C$  is almost Dedekind and  $M = qC$ , we can write  $IC = M^i J$  where  $J$  is an ideal of  $C$  with  $M + J = C$  and  $i \geq 0$ , so  $IC = M^i \cap J$ . We also see that  $H := J \cap A \not\subseteq M$ . As  $\cap_{t \geq 1} p^t A = M$ , we can write  $H = p^j L = p^j A \cap L$  where  $L$  is an ideal of  $A$  with  $L \not\subseteq pA$  and  $j \geq 0$ . Consequently we get

$$IC \cap A = M^i \cap J \cap A = M^i \cap H = M^i \cap p^j A \cap L$$

with equals either  $M^i \cap L$  if  $i \geq 1$  or  $p^j A \cap L$  if  $i = 0$ . Using basic facts on valuation domains (see [8, Section 17]), it suffices to consider the following three cases. Each time we use the equality  $I = (IB \cap A) \cap (IC \cap A)$ .

Case 1:  $IB = p^n B$  for some  $n \geq 0$ . We have  $IB \cap A = p^n A$ . If  $i \geq 1$ , we get  $I = p^n A \cap M^i \cap L = M^i L$ . If  $i = 0$ , we get  $I = p^n A \cap p^j A \cap L = p^k L$  with  $k = \max(n, j)$ .

Case 2:  $IB = M^n$  for some  $n \geq 1$ . If  $i \geq 1$ , we get  $I = M^n \cap M^i \cap L = M^k L$  with  $k = \max(n, i)$ . If  $i = 0$ , we get  $I = M^n \cap p^j A \cap L = M^n L$ .

Case 3:  $IB = p^n q^m A$  for some  $m \geq 1$  and  $n \in \mathbb{Z}$ . We have  $IB \cap A = p^n q^m A$ , because  $pA$  is the only maximal ideal containing  $q$ . If  $i > m \geq 1$ , we get  $I =$

$p^n q^m A \cap M^i \cap L = M^i L$ . If  $m \geq i \geq 1$ , we get  $I = p^n q^m A \cap M^i \cap L = p^n q^m L$ . If  $i = 0$ , we get  $I = p^n q^m A \cap p^j A \cap L = p^n q^m L$ .

Consequently, to complete our proof, it suffices to show that  $L$  is a product of radical ideals. Since  $C$  is an SP-domain, we can write  $LC = H_1 \cdots H_n$  with each  $H_i$  a radical ideal of  $C$ . Then each  $J_i = H_i \cap A$  is a radical ideal of  $A$ . Note that none of ideals  $J_i$  is contained in  $pA$ , since  $L \not\subseteq pA$ . Set  $R = J_1 \cdots J_n$ . Then  $R + pA = A$  and  $L + pA = A$ , so  $R : p = R$  and  $L : p = L$ . Since  $RC = H_1 \cdots H_n = LC$ , we get  $L = LC \cap A = RC \cap A = R$ .  $\square$

Finally, we construct a specific domain satisfying the hypothesis of Proposition 25. We modify appropriately [6, Example 3.4.1]. If  $A$  is a domain and  $P_1, \dots, P_n$  are prime ideals of  $A$ , we denote by  $A_{P_1 \cup \dots \cup P_n}$  the fraction ring of  $A$  with denominators in  $A - (P_1 \cup \dots \cup P_n)$ . Let  $y$  and  $(x_n)_{n \geq 1}$  be indeterminates over the rational field  $\mathbb{Q}$ . Consider the domain

$$C = \bigcup_{n \geq 1} \mathbb{Q}[x_1, \dots, x_n, y/(x_1 \cdots x_n)]_{(x_1) \cup \dots \cup (x_n) \cup (y/(x_1 \cdots x_n))}.$$

As  $C$  is a union of an ascending chain of (semi-local) PID's, it is a one-dimensional Bezout domain. Adapting the proof of [6, Example 3.4.1], we see that the maximal ideals of  $C$  are  $N = \sum_{n \geq 1} (y/(x_1 \cdots x_n))C$  and the principal ideals  $(x_n C)_{n \geq 1}$ . As  $yC_M = MC_M$  for each  $M \in \text{Max}(C)$ , it follows that  $yC$  is a radical ideal, hence  $N$  is non-critical. By [10, Corollary 2.2],  $C$  is an SP-domain. The residue field  $C/x_1 C$  is isomorphic to  $K(y/x_1)$  where  $K = \mathbb{Q}(x_n; n \geq 2)$ . Then  $D = K[y/x_1]_{(y/x_1)}$  is a DVR with quotient field  $C/x_1 C$ . Note that  $x_1 + y/x_1$  is a unit of  $\mathbb{Q}[x_1, y/x_1]_{(x_1) \cup (y/x_1)}$ , hence a unit of  $C$ . Moreover, the canonical map  $C \rightarrow C/x_1 C$  sends  $x_1 + y/x_1$  to  $y/x_1$  which is a generator of the maximal ideal of  $D$ . Thus  $C$  satisfies the hypothesis of Proposition 25.

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